

GREEDY AND QUASI-GREEDY EXPANSIONS IN NON-INTEGERS BASES

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ABSTRACT. We generalize several theorems of Rényi, Parry, Daróczy and Kátai by characterizing the greedy and quasi-greedy expansions in non-integer bases.

1. INTRODUCTION

Fix a positive integer M . By a *sequence* we mean a sequence $(c_i) = c_1c_2\ldots$ satisfying $c_i \in \{0, 1, \dots, M\}$ for each i . A sequence is called *finite* if it contains only finitely many nonzero terms; otherwise it is called *infinite*.

Given a real number $q > 1$ and a nonnegative real number x , by an *expansion* of x we mean a sequence (c_i) satisfying

$$\frac{c_1}{q} + \frac{c_2}{q^2} + \cdots = x.$$

This can only happen if $x \in [0, M/(q-1)]$ because

$$0 \leq \frac{c_1}{q} + \frac{c_2}{q^2} + \cdots \leq \frac{M}{q} + \frac{M}{q^2} + \cdots = \frac{M}{q-1}.$$

If $q \leq M+1$, then the converse statement also holds: every $x \in [0, M/(q-1)]$ has at least one expansion. More precisely, we will show that every $x \in [0, M/(q-1)]$ has a lexicographically largest expansion and every $x \in (0, M/(q-1)]$ has a lexicographically largest *infinite* expansion. The *lexicographical* order between sequences is defined in the usual way: we write

$$a < b, \quad (a_i) < (b_i) \quad \text{or} \quad a_1a_2\ldots < b_1b_2\ldots$$

if there exists an index $n \geq 1$ such that

$$a_1 \ldots a_{n-1} = b_1 \ldots b_{n-1} \quad \text{and} \quad a_n < b_n.$$

Furthermore, we write

$$a \leq b, \quad (a_i) \leq (b_i) \quad \text{or} \quad a_1a_2\ldots \leq b_1b_2\ldots$$

if we also allow the equality of the two sequences.

We will give two simple algorithms for the construction of these special expansions, we will characterize them algebraically, and we will compare them. Finally, we also study the limiting case $M = \infty$ when every sequence of nonnegative integers is admitted as a possible expansion.

The results of this note extend various former theorems due to Rényi [9], Parry [7], Daróczy and Kátai [1] (see also [4]) and they have been already applied in the study of unique expansions [5], [2] and of expansions with deleted digits by Pedicini [8].

2. QUASI-GREEDY EXPANSIONS

Fix a positive integer M and a real number $q > 1$. If $(c_i) = c_1 c_2 \dots$ is an infinite expansion of x , then

$$0 < \frac{c_1}{q} + \frac{c_2}{q^2} + \dots \leq \frac{M}{q} + \frac{M}{q^2} + \dots = \frac{M}{q-1},$$

so that $x \in (0, M/(q-1)]$.

In order to prove a converse statement, let us introduce for each $x > 0$ the lexicographically largest infinite sequence (a_i) satisfying

$$(2.1) \quad \frac{a_1}{q} + \frac{a_2}{q^2} + \dots \leq x.$$

This is equivalent to the following recursive definition: if a_k has already been defined for all $k < n$ (no assumption if $n = 1$), then let a_n be the largest integer satisfying the inequalities

$$a_n \leq M \quad \text{and} \quad \frac{a_1}{q} + \dots + \frac{a_n}{q^n} < x.$$

Since $x > 0$, the definition is correct. We have the following

Proposition 2.1. *If $M \geq q-1$ and $0 < x \leq M/(q-1)$, then (a_i) is an infinite expansion of x .*

Proof. In view of (2.1) it suffices to establish the converse inequality

$$(2.2) \quad \frac{a_1}{q} + \frac{a_2}{q^2} + \dots \geq x.$$

By definition we have

$$x - \frac{1}{q^n} \leq \frac{a_1}{q} + \dots + \frac{a_n}{q^n} \quad \text{whenever} \quad a_n < M.$$

If there are infinitely many such indices then letting $n \rightarrow \infty$ hence (2.2) follows.

If $a_i = M$ for all i , then

$$\frac{a_1}{q} + \frac{a_2}{q^2} + \dots = \frac{M}{q-1} \geq x$$

by our assumption on x .

Finally, if there exists a last digit $a_n < M$, then

$$\begin{aligned} \frac{a_1}{q} + \frac{a_2}{q^2} + \dots &= \frac{a_1}{q} + \dots + \frac{a_n}{q^n} + \frac{M}{(q-1)q^n} \\ &\geq x - \frac{1}{q^n} + \frac{M}{(q-1)q^n} \\ &\geq x \end{aligned}$$

by our assumption $M \geq q-1$. □

Definition. If $M \geq q-1$ and $0 < x \leq M/(q-1)$, then (a_i) is called the *quasi-greedy* expansion of x . (The terminology will be clarified in the next section.)

Remark. Since it is the lexicographically largest infinite sequence satisfying (2.1), the quasi-greedy expansion is the lexicographically largest infinite expansion.

One can recognize the quasi-greedy expansions by their form:

Theorem 2.2.

(a) *The map $q \mapsto (\alpha_i)$, where (α_i) denotes the quasi-greedy expansion of 1, is a strictly increasing one-to-one correspondence between the interval $(1, M+1]$ and the set of infinite sequences satisfying*

$$(2.3) \quad \alpha_{n+1}\alpha_{n+2}\dots \leq \alpha_1\alpha_2\dots \quad \text{whenever} \quad \alpha_n < M.$$

(b) Fix $q \in (1, M + 1]$ arbitrarily and denote by (α_i) the quasi-greedy expansion of 1. The map $x \mapsto (a_i)$, where (a_i) denotes the quasi-greedy expansion of x , is a strictly increasing one-to-one correspondence between the interval $(0, M/(q - 1)]$ and the set of infinite sequences satisfying

$$(2.4) \quad a_{n+1}a_{n+2} \dots \leq \alpha_1\alpha_2 \dots \quad \text{whenever} \quad a_n < M.$$

For the proof we need the following result:

Lemma 2.3. *Let (α_i) be an arbitrary expansion of 1. If a sequence (a_i) satisfies the condition*

$$a_{n+1}a_{n+2} \dots \leq \alpha_1\alpha_2 \dots \quad \text{whenever} \quad a_n < M,$$

then we also have

$$\frac{a_{n+1}}{q^{n+1}} + \frac{a_{n+2}}{q^{n+2}} + \dots \leq \frac{1}{q^n}$$

whenever $a_n < M$.

Consequently, if the sequence (a_i) is also infinite, it is the quasi-greedy expansion of

$$x := \frac{a_1}{q} + \frac{a_2}{q^2} + \dots.$$

Proof. Starting with $k_0 := n$ let us define by recurrence a sequence of indices $k_0 < k_1 < \dots$ satisfying for $j = 1, 2, \dots$ the conditions

$$a_{k_{j-1}+i} = \alpha_i \quad \text{for} \quad i = 1, \dots, k_j - k_{j-1} - 1, \quad \text{and} \quad a_{k_j} < \alpha_{k_j - k_{j-1}}.$$

If we obtain an infinite sequence, then we have

$$\begin{aligned} \sum_{i=n+1}^{\infty} \frac{a_i}{q^i} &= \sum_{j=1}^{\infty} \sum_{i=1}^{k_j - k_{j-1}} \frac{a_{k_{j-1}+i}}{q^{k_{j-1}+i}} \\ &\leq \sum_{j=1}^{\infty} \left(\left(\sum_{i=1}^{k_j - k_{j-1}} \frac{\alpha_i}{q^{k_{j-1}+i}} \right) - \frac{1}{q^{k_j}} \right) \\ &\leq \sum_{j=1}^{\infty} \left(\frac{1}{q^{k_{j-1}}} - \frac{1}{q^{k_j}} \right) \\ &= \frac{1}{q^n}. \end{aligned}$$

Otherwise we have $(a_{k_N+i}) = (\alpha_i)$ after a finite number of steps (we do not exclude the possibility that $N = 0$), and we may conclude as follows:

$$\begin{aligned} \sum_{i=n+1}^{\infty} \frac{a_i}{q^i} &= \left(\sum_{j=1}^N \sum_{i=1}^{k_j - k_{j-1}} \frac{a_i}{q^{k_{j-1}+i}} \right) + \sum_{i=1}^{\infty} \frac{a_{k_N+i}}{q^{k_N+i}} \\ &\leq \sum_{j=1}^N \left(\left(\sum_{i=1}^{k_j - k_{j-1}} \frac{\alpha_i}{q^{k_{j-1}+i}} \right) - \frac{1}{q^{k_j}} \right) + \sum_{i=1}^{\infty} \frac{\alpha_i}{q^{k_N+i}} \\ &\leq \sum_{j=1}^N \left(\frac{1}{q^{k_{j-1}}} - \frac{1}{q^{k_j}} \right) + \frac{1}{q^{k_N}} \\ &= \frac{1}{q^n}. \end{aligned} \quad \square$$

If the sequence (a_i) is infinite, then setting

$$x := \frac{a_1}{q} + \frac{a_2}{q^2} + \dots$$

our result can be written in the form

$$\frac{a_1}{q} + \cdots + \frac{a_n}{q^n} \geq x - \frac{1}{q^n}$$

whenever $a_n < M$, and this proves that (a_i) is the quasi-greedy expansion of x .

Proof of Theorem 2.2. The strict increasingness of both maps follows from the definition of quasi-greedy expansions.

In order to prove that every quasi-greedy expansion satisfies (2.4) it suffices to observe that if $a_n < M$ for some n , then we infer from the inequalities

$$\frac{a_1}{q} + \cdots + \frac{a_{n-1}}{q^{n-1}} + \frac{a_n + 1}{q^n} \geq x = \frac{a_1}{q} + \frac{a_2}{q^2} + \cdots$$

that

$$\frac{a_{n+1}}{q} + \frac{a_{n+2}}{q^2} + \cdots \leq 1.$$

This yields (2.4) because (α_i) is by definition the lexicographically largest infinite sequence satisfying such an inequality (see (2.1)). The condition (2.3) hence follows by taking $x = 1$.

Finally, the *onto* property of both maps follows from the preceding lemma. \square

Remarks.

- The results of this section extend to the case $M = \infty$, i.e., when all non-negative digits are permitted in the expansions. In this case the conditions $M \geq q-1$, $x \leq M/(q-1)$, $a_n < M$ and $\alpha_n < M$ are automatically satisfied, and hence can be omitted.
- Observe that if $0 < x \leq (M+1)/q$, then the quasi-greedy expansion of x remains the same by changing M to infinity. Consequently, in Theorem 2.2 the condition (2.3) is satisfied for *all* n , and (2.4) is also satisfied for *all* n if $0 < x \leq (M+1)/q$.

For the sake of convenience we end this section by giving explicitly the results corresponding to the case $M = \infty$. We fix a real number $q > 1$.

Definition. The *quasi-greedy expansion* of a positive real number x is by definition the lexicographically largest infinite sequence (a_i) of nonnegative integers satisfying

$$\frac{a_1}{q} + \frac{a_2}{q^2} + \cdots \leq x.$$

Equivalently, the sequence (a_i) is defined recursively as follows: if a_k has already been defined for all $k < n$ (no assumption if $n = 1$), then a_n is the largest integer satisfying the inequality

$$\frac{a_1}{q} + \cdots + \frac{a_n}{q^n} < x.$$

We have the following

Proposition 2.4. *If $x > 0$, then (a_i) is an infinite expansion of x , i.e., it has infinitely many nonzero elements and*

$$\frac{a_1}{q} + \frac{a_2}{q^2} + \cdots = x.$$

Furthermore, we have $a_n < q$ for all $n \geq 2$.

Proof. We have

$$\frac{a_1}{q} + \cdots + \frac{a_n}{q^n} < x \leq \frac{a_1}{q} + \cdots + \frac{a_n}{q^n} + \frac{1}{q^n}$$

for all $n \geq 1$ by definition. Hence

$$0 < x - \left(\frac{a_1}{q} + \cdots + \frac{a_n}{q^n} \right) \leq \frac{1}{q^n}$$

for all n and the right-hand side converges to zero.

It follows from the inequalities

$$\frac{a_1}{q} + \cdots + \frac{a_n}{q^n} + \frac{a_{n+1}}{q^{n+1}} < x \leq \frac{a_1}{q} + \cdots + \frac{a_n}{q^n} + \frac{1}{q^n}$$

that

$$\frac{a_{n+1}}{q^{n+1}} < \frac{1}{q^n}$$

and therefore $a_{n+1} < q$ for $n = 1, 2, \dots$ □

Theorem 2.5.

(a) The map $q \mapsto (\alpha_i)$, where (α_i) denotes the quasi-greedy expansion of 1, is a strictly increasing one-to-one correspondence between the interval $(1, \infty)$ and the set of infinite sequences satisfying

$$\alpha_{n+1}\alpha_{n+2}\dots \leq \alpha_1\alpha_2\dots \quad \text{for all } n.$$

(b) Fix $q > 1$ arbitrarily and denote by (α_i) the quasi-greedy expansion of 1. The map $x \mapsto (a_i)$, where (a_i) denotes the quasi-greedy expansion of x , is a strictly increasing one-to-one correspondence between the interval $(0, \infty)$ and the set of infinite sequences satisfying

$$a_{n+1}a_{n+2}\dots \leq \alpha_1\alpha_2\dots \quad \text{for all } n.$$

Remark. It follows from Propositions 2.1 and 2.4 that the quasi-greedy expansion of x is the same for all M satisfying $0 < x \leq (M+1)/q$ (including $M = \infty$). In particular, the quasi-greedy expansion (α_i) of $x = 1$ is the same for all $q-1 \leq M \leq \infty$. Consequently, in Theorem 2.2 the condition (2.3) is satisfied for all n , and (2.4) is also satisfied for all n if $0 < x \leq (M+1)/q$.

Proof. We may repeat the proof of Lemma 2.3 (by simply omitting the words “whenever $a_n < M$ ” in its statement) and Theorem 2.2. □

3. GREEDY EXPANSIONS

Fix a positive integer M and a real number $q > 1$. We recall that x must belong to the interval $[0, M/(q-1)]$ in order to have an expansion.

In order to prove a converse statement, let us introduce for each $x \geq 0$ the lexicographically largest sequence (b_i) satisfying

$$(3.1) \quad \frac{b_1}{q} + \frac{b_2}{q^2} + \cdots \leq x.$$

This is equivalent to the following recursive definition: if b_k has already been defined for all $k < n$ (no assumption if $n = 1$), then let b_n be the largest integer satisfying the inequalities

$$b_n \leq M \quad \text{and} \quad \frac{b_1}{q} + \cdots + \frac{b_n}{q^n} \leq x.$$

Since $x \geq 0$, the definition is correct. First we prove the following variant of Proposition 2.1:

Proposition 3.1. *If $M \geq q-1$ and $0 \leq x \leq M/(q-1)$, then (b_i) is an expansion of x .*

Proof. The case $x = 0$ is obvious: then (b_i) is the null sequence.

If $x > 0$, then comparing with the recursive definition of the quasi-greedy expansions we obtain that

$$\frac{a_1}{q} + \frac{a_2}{q^2} + \cdots \leq \frac{b_1}{q} + \frac{b_2}{q^2} + \cdots \leq x$$

for all n . If $M \geq q - 1$ and $x \leq M/(q - 1)$, then the left-hand side tends to x by Proposition 2.1 and therefore (b_i) is also an expansion of x . This is obviously satisfied for $x = 0$, too, when (b_i) is the null sequence. \square

Definition. If $M \geq q - 1$ and $0 \leq x \leq M/(q - 1)$, then (b_i) is called the *greedy expansion* of x .

Remark. As the lexicographically largest sequence satisfying (3.1), the greedy expansion of x is the lexicographically largest expansion of x .

Next we prove a variant of Theorem 2.2. It is convenient to define the greedy expansion of 1 for $q = 1$ by setting $(\beta_i) = 10^\infty$, i.e., $\beta_1 = 1$ and $\beta_i = 0$ for all $i > 1$.

Theorem 3.2.

(a) The map $q \mapsto (\beta_i)$, where (β_i) denotes the greedy expansion of 1, is a strictly increasing one-to-one correspondence between the interval $[1, M + 1]$ and the set of all sequences satisfying

$$(3.2) \quad \beta_{n+1}\beta_{n+2}\dots < \beta_1\beta_2\dots \quad \text{whenever} \quad \beta_n < M.$$

(b) Fix $1 < q \leq M + 1$ arbitrarily and denote by (α_i) the quasi-greedy expansion of 1. The map $x \mapsto (b_i)$, where (b_i) denotes the greedy expansion of x , is a strictly increasing one-to-one correspondence between the interval $[0, M/(q - 1)]$ and the set of all sequences satisfying

$$(3.3) \quad b_{n+1}b_{n+2}\dots < \alpha_1\alpha_2\dots \quad \text{whenever} \quad b_n < M.$$

Remark. Part (b) of this theorem is a slight generalization of earlier theorems obtained by Parry [7] and Daróczy and Kátai [1].

We need a variant of Lemma 2.3:

Lemma 3.3. If an expansion (β_i) of 1 for some $q \geq 1$ satisfies the condition

$$\beta_{n+1}\beta_{n+2}\dots < \beta_1\beta_2\dots \quad \text{whenever} \quad \beta_n < M,$$

then it is the greedy expansion of 1.

Proof. The case $q = 1$ is obvious, so we assume henceforth that $q > 1$. Defining the sequence $k_0 < k_1 < \dots$ as in the proof of Lemma 2.3, now we always obtain an infinite sequence, and hence

$$\begin{aligned} \sum_{i=n+1}^{\infty} \frac{\beta_i}{q^i} &= \sum_{j=1}^{\infty} \sum_{i=1}^{k_j - k_{j-1}} \frac{\beta_{k_{j-1}+i}}{q^{k_{j-1}+i}} \\ &\leq \sum_{j=1}^{\infty} \left(\left(\sum_{i=1}^{k_j - k_{j-1}} \frac{\beta_i}{q^{k_{j-1}+i}} \right) - \frac{1}{q^{k_j}} \right) \\ &\leq \sum_{j=1}^{\infty} \left(\frac{1}{q^{k_{j-1}}} - \frac{1}{q^{k_j}} \right) \\ &= \frac{1}{q^n} \end{aligned}$$

whenever $\beta_n < M$. It remains to exclude the equality here. In the last computation we have equality only if (β_i) has a last nonzero term β_m and if the sequence (β_i) is periodic with period $\beta_1 \dots \beta_{m-1} \beta_m^-$, $\beta_m^- = \beta_m - 1$. Since then (β_i) is a finite sequence, this implies $m = 1$ and $\beta_m = 1$. However, this corresponds to the case $q = 1$, already excluded. \square

Proof of the theorem. The strict increasingness of both maps follows from the definition of greedy expansions. They are onto by Lemmas 2.3 and 3.3. It remains to show that every greedy expansion satisfies the inequalities (3.2) and (3.3), respectively.

For the proof of (3.2) it suffices to observe that if $\beta_n < M$ for some n , then we infer from the inequalities

$$\frac{\beta_1}{q} + \cdots + \frac{\beta_{n-1}}{q^{n-1}} + \frac{\beta_n + 1}{q^n} > 1 = \frac{\beta_1}{q} + \frac{\beta_2}{q^2} + \cdots$$

that

$$\frac{\beta_{n+1}}{q} + \frac{\beta_{n+2}}{q^2} + \cdots < 1.$$

This shows first that $(\beta_{n+i}) \neq (\beta_i)$ because for the latter sequence we have equality, and secondly that $(\beta_{n+i}) \leq (\beta_i)$ because (β_i) is by definition the lexicographically largest sequence satisfying

$$\frac{\beta_1}{q} + \frac{\beta_2}{q^2} + \cdots \leq 1.$$

Therefore $(\beta_{n+i}) < (\beta_i)$.

The beginning of the proof of (3.3) is similar: if $b_n < M$ for some n , then we infer from the inequalities

$$\frac{b_1}{q} + \cdots + \frac{b_{n-1}}{q^{n-1}} + \frac{b_n + 1}{q^n} > x = \frac{b_1}{q} + \frac{b_2}{q^2} + \cdots$$

that

$$\frac{b_{n+1}}{q} + \frac{b_{n+2}}{q^2} + \cdots < 1.$$

This shows first that $(b_{n+i}) \neq (\alpha_i)$ because for the latter sequence we have equality. Next, if the sequence (b_i) is infinite, then it also shows that $(b_{n+i}) \leq (\alpha_i)$ because (α_i) is by definition the lexicographically largest infinite sequence satisfying

$$\frac{\alpha_1}{q} + \frac{\alpha_2}{q^2} + \cdots \leq 1.$$

Therefore $(b_{n+i}) < (\alpha_i)$.

For $x = 0$ this condition is obviously satisfied, too. In case (b_i) has a last nonzero digit b_m we may apply the above argument to the infinite expansion of x with period $b_1 \dots b_{m-1} b_m^-$ where $b_m^- = b_m - 1$, to obtain

$$(3.4) \quad (b_1 \dots b_{m-1} b_m^-)^\infty < (\alpha_i)$$

with obvious notation. If (3.3) were not satisfied, then we would infer from the relations

$$b_1 \dots b_{m-1} b_m^- \leq \alpha_1 \dots \alpha_m < b_1 \dots b_{m-1} b_m$$

that

$$\alpha_1 \dots \alpha_{m-1} \alpha_m = b_1 \dots b_{m-1} b_m^-$$

and therefore (3.4) may be rewritten as

$$(\alpha_1 \dots \alpha_{m-1} \alpha_m)^\infty < (\alpha_i).$$

However this is impossible because $(\alpha_{n+i}) \leq (\alpha_i)$ for every n by condition (2.3) of Theorem 2.2 and by the remark at the end of the preceding section, and hence

$$(\alpha_i) \leq (\alpha_1 \dots \alpha_{m-1} \alpha_m)^\infty. \quad \square$$

Next we clarify the relations between quasi-greedy and greedy expansions:

Proposition 3.4. *Given $1 < q \leq M + 1$ and $0 < x \leq M/(q - 1)$, let us denote by (a_i) , (b_i) the quasi-greedy and greedy expansions of x , and by (α_i) , (β_i) those of 1.*

(a) *If (b_i) has a last nonzero element b_m , then*

$$(a_i) = b_1 \dots b_{m-1} b_m^- \alpha_1 \alpha_2 \dots$$

with $b_m^- = b_m - 1$. Otherwise we have $(a_i) = (b_i)$.

(b) *If (β_i) has a last nonzero element β_m , then (α_i) is periodic with the smallest period $\beta_1 \dots \beta_{m-1} \beta_m^-$ where $\beta_m^- = \beta_m - 1$, i.e.,*

$$(\alpha_i) = (\beta_1 \dots \beta_{m-1} \beta_m^-)^\infty.$$

Otherwise we have $(\beta_i) = (\alpha_i)$, and this sequence is periodic only in the extreme case $q = M + 1$ when $(\beta_i) = (\alpha_i) = M^\infty$.

Proof.

(a) The only nontrivial property is that if (b_i) has a last nonzero element b_m , then $(c_i) \leq b_1 \dots b_{m-1} b_m^- \alpha_1 \alpha_2 \dots$ for every infinite expansion of x . Since $(c_i) < (b_i)$, we must have $c_1 \dots c_m \leq b_1 \dots b_{m-1} b_m^-$. If we have equality here, then $c_{m+1} c_{m+2} \dots$ is an infinite expansion of 1, so that $c_{m+1} c_{m+2} \dots \leq \alpha_1 \alpha_2 \dots$.

(b) If (β_i) has a last nonzero element β_m , then $(\beta_1 \dots \beta_{m-1} \beta_m^-)^\infty$ is clearly an infinite expansion of 1. (Observe that the case $m = 1$ and $\beta_1 = 1$ is excluded by our assumption $q > 1$.) If (γ_i) is an infinite expansion of 1, then $\gamma_1 \dots \gamma_m < \beta_1 \dots \beta_m$, so that $\gamma_1 \dots \gamma_m \leq \beta_1 \dots \beta_{m-1} b_m^-$. In case of equality $\gamma_{m+1} \gamma_{m+2} \dots$ is again an infinite expansion of 1, so that $\gamma_{m+1} \gamma_{m+2} \dots \leq \beta_1 \dots \beta_{m-1} b_m^-$. Iterating this reasoning we find that $(\gamma_i) \leq (\beta_1 \dots \beta_{m-1} \beta_m^-)^\infty$. This proves that $(\beta_1 \dots \beta_{m-1} \beta_m^-)^\infty$ is the largest infinite expansion of 1.

The latter sequence cannot have a shorter period of length $k < m$ because then k should divide m and condition (3.2) would be violated for $n = m - k$: putting $\beta_k^+ = \beta_k + 1$ we would have

$$\beta_{m-k+1} \dots \beta_m = \beta_1 \dots \beta_{k-1} \beta_k^+ > \beta_1 \dots \beta_k.$$

Finally, if $(\beta_i) = (\alpha_i)$ has a smallest period $\beta_1 \dots \beta_m$, then $\beta_m = M$, for otherwise $\beta_1 \dots \beta_{m-1} \beta_m^+ 0^\infty$ would be a larger expansion of 1. Then applying (2.3) for $(\beta_i) = (\alpha_i)$ we have

$$\alpha_m \alpha_{m+1} \dots \alpha_{2m-1} \leq \alpha_1 \alpha_2 \dots \alpha_m,$$

i.e.,

$$M \alpha_1 \dots \alpha_{m-1} \leq \alpha_1 \alpha_2 \dots \alpha_m,$$

This yields successively $\alpha_1 = M$, $\alpha_2 = M$, \dots , $\alpha_{m-1} = M$, so that $(\beta_i) = (\alpha_i) = M^\infty$. \square

Remarks.

- The proposition and its proof allow us to determine all expansions of x between (a_i) and (b_i) :
 - If (b_i) is infinite, then $(a_i) = (b_i)$.
 - If (b_i) has a last nonzero element b_m and if (β_i) is infinite, then $(a_i) = b_1 \dots b_{m-1} b_m^- \beta_1 \beta_2 \dots$ is the second largest expansion of x .
 - If both (b_i) and (β_i) have last nonzero element b_m and β_n , then the expansions (c_i) of x satisfying $(a_i) < (c_i) < (b_i)$ are given by the decreasing sequence

$$b_1 \dots b_{m-1} b_m^- (\beta_1 \dots \beta_{n-1} \beta_n^-)^N (\beta_1 \dots \beta_n) 0^\infty, \quad N = 0, 1, \dots$$

- The results of this section extend to $M = \infty$, too. Then the conditions $M \geq q - 1$, $x \leq M/(q - 1)$, $b_n < M$ and $\beta_n < M$ may be omitted. The greedy expansions in case $M = \infty$ are exactly the beta-expansions introduced by Rényi [9].

- If $0 \leq x < (M+1)/q$, then the greedy expansion of x does not change by changing M to infinity. Therefore condition (3.2) of Theorem 3.2 is satisfied for all n , and condition (3.3) is satisfied for all n if $0 \leq x \leq (M+1)/q$ (the obvious cases $M = q-1$ and $x = (M+1)/q$ can be checked separately).

We end this paper by giving again the explicit statements concerning the case $M = \infty$.

Definition. The *greedy expansion* of a nonnegative real number x is by definition the lexicographically largest sequence (b_i) of nonnegative integers satisfying

$$\frac{b_1}{q} + \frac{b_2}{q^2} + \cdots \leq x.$$

Equivalently, the sequence (b_i) is defined recursively as follows: if b_k has already been defined for all $k < n$ (no assumption if $n = 1$), then b_n is the largest integer satisfying the inequality

$$\frac{b_1}{q} + \cdots + \frac{b_n}{q^n} \leq x.$$

We have the following

Proposition 3.5. *If $x \geq 0$, then (b_i) is an expansion of x , i.e.,*

$$\frac{b_1}{q} + \frac{b_2}{q^2} + \cdots = x.$$

Furthermore, we have $b_n < q$ for all $n \geq 2$.

Proof. We have

$$\frac{b_1}{q} + \cdots + \frac{b_n}{q^n} \leq x < \frac{b_1}{q} + \cdots + \frac{b_n}{q^n} + \frac{1}{q^n}$$

for all $n \geq 1$ by definition. Hence

$$0 \leq x - \left(\frac{b_1}{q} + \cdots + \frac{b_n}{q^n} \right) < \frac{1}{q^n}$$

for all n and the right-hand side converges to zero.

It follows from the inequalities

$$\frac{b_1}{q} + \cdots + \frac{b_n}{q^n} + \frac{b_{n+1}}{q^{n+1}} \leq x < \frac{b_1}{q} + \cdots + \frac{b_n}{q^n} + \frac{1}{q^n}$$

that

$$\frac{b_{n+1}}{q^{n+1}} < \frac{1}{q^n}$$

and therefore $b_{n+1} < q$ for $n = 1, 2, \dots$ □

The following result is essentially due to Parry [7]; following Daróczy and Kátai [1] we simplify its formulation by using quasi-greedy expansions. It is convenient to introduce again the greedy expansion of 1 for $q = 1$ by setting $(\beta_i) = 10^\infty$, i.e., $\beta_1 = 1$ and $\beta_i = 0$ for all $i > 1$.

Theorem 3.6.

(a) *The map $q \mapsto (\beta_i)$, where (β_i) denotes the greedy expansion of 1, is a strictly increasing one-to-one correspondence between the interval $[1, \infty)$ and the set of all sequences satisfying*

$$\beta_{n+1}\beta_{n+2}\cdots < \beta_1\beta_2\cdots \quad \text{for all } n.$$

(b) *Fix $q > 1$ arbitrarily and denote by (α_i) the quasi-greedy expansion of 1. The map $x \mapsto (b_i)$, where (b_i) denotes the greedy expansion of x , is a strictly increasing*

one-to-one correspondence between the interval $[0, \infty)$ and the set of all sequences satisfying

$$b_{n+1}b_{n+2}\dots < \alpha_1\alpha_2\dots \quad \text{for all } n.$$

Proof. We may repeat the proof of Lemma 3.3 (by changing the words “whenever $\beta_n < M$ ” to “for all $n \geq 1$ ”) and Theorem 3.2 (by changing the words “if $\beta_n < M$ ” to “for all $n \geq 1$ ”). \square

Next we clarify the relations between quasi-greedy and greedy expansions:

Proposition 3.7. *Given $q > 1$ and $x > 0$, let us denote by (a_i) , (b_i) the quasi-greedy and greedy expansions of x , and by (α_i) , (β_i) those of 1.*

(a) *If (b_i) has a last nonzero element b_m , then*

$$(a_i) = b_1 \dots b_{m-1} b_m^- \alpha_1 \alpha_2 \dots$$

with $b_m^- = b_m - 1$. Otherwise we have $(a_i) = (b_i)$.

(b) *If (β_i) has a last nonzero element β_m , then (α_i) is periodic with the smallest period $\beta_1 \dots \beta_{m-1} \beta_m^-$ where $\beta_m^- = \beta_m - 1$, i.e.,*

$$(\alpha_i) = (\beta_1 \dots \beta_{m-1} \beta_m^-)^\infty.$$

Otherwise we have $(\beta_i) = (\alpha_i)$, and this sequence is not periodic.

Proof. We may repeat the proof of Proposition 3.4, by shortening its last paragraph by observing that if $(\beta_i) = (\alpha_i)$ had a period of length m , then $\beta_1 \dots \beta_{m-1} \beta_m^+ 0^\infty$ would also be an expansion of 1, contradicting the maximality of (β_i) . \square

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